Critical exponents of the anisotropic Bak-Sneppen model

Sergei Maslov,[1] Paolo De Los Rios,[2] Matteo Marsili,[2,3] and Yi-Cheng Zhang[2]
[1]*Department of Physics, Brookhaven National Laboratory, Upton, New York 11973*
[2]*Institut de Physique Théorique, Université de Fribourg Pérolles, Fribourg CH-1700, Switzerland*
[3]*International School for Advanced Studies (SISSA) and INFM Unit, Trieste I-34014, Italy*
(Received 26 March 1998; revised manuscript received 27 August 1998)

We analyze the behavior of the spatially anisotropic Bak-Sneppen model. We demonstrate that a nontrivial relation between critical exponents τ and $\mu = d/D$, recently derived for the isotropic Bak-Sneppen model, holds for its anisotropic version as well. For the one-dimensional anisotropic Bak-Sneppen model, we derive an exact equation for the distribution of avalanche spatial sizes, and extract the value $\gamma = 2$ for one of the critical exponents of the model. Other critical exponents are then determined from previously known exponent relations. Our results are in excellent agreement with Monte Carlo simulations of the model as well as with direct numerical integration of the new equation. [S1063-651X(98)13512-2]

PACS number(s): 05.40.+j, 64.60.Ak, 64.60.Fr, 87.10.+e

Ever since its introduction five years ago, the Bak-Sneppen (BS) model [1] was a subject of considerable theoretical interest. Its relevance relies on the fact that it provides a very simple mechanism of self-organized criticality [2]. In fact, the Bak-Sneppen model is the simplest representative of a broad class of extremal models, which all naturally evolve towards a scale-free stationary state [3]. An extremely rich dynamic critical behavior arising out of truly minimalistic dynamical rules has inspired numerous analytical and numerical investigations of the Bak-Sneppen model [3–10].

In this work we introduce and study an *anisotropic* version of the original Bak-Sneppen model. In one dimension the dynamics is as follows: the configuration of the system is fully defined by the value of variable f_i for each lattice site i. At every time step the smallest variable in the system and that at its *right* nearest neighbor are replaced with new random numbers independently drawn from the distribution $\mathcal{P}(f) = e^{-f}$ [11]. Contrary to the isotropic BS model, where both nearest neighbor variables are updated, only the variable in the preferred direction is updated here. Other mechanisms of introducing spatial anisotropy to the rules of the original BS model were recently studied in Refs. [12,13]. As we shall see, the universality of the critical behavior manifests itself in the fact that any realization of the anisotropy in the original BS model gives rise to the same set of critical exponents [12]. The generalization of our version of anisotropic BS model to higher dimensions is straightforward: only d neighbors of the global minimal site located in positive directions of corresponding coordinates are updated along with it.

This work is devoted to analytical and numerical study of exponents of the anisotropic Bak-Sneppen model. The main observations used in the analytical part of this study are as follows. (i) The general scaling theory of Ref. [3] developed for an arbitrary extremal model reduces the number of independent critical exponents to just two. (ii) The relation $\tau(\mu)$ between two remaining exponents τ and $\mu = d/D$, recently derived in [8,9] for the isotropic Bak-Sneppen model in an arbitrary dimension, holds for its anisotropic version as well. Finally, (iii) in the one-dimensional anisotropic BS model the exponent $\gamma = 2$, describing the divergence of the average avalanche size, can be derived from the *exact* master equation for the probability distribution of avalanche spatial sizes. This equation, which we derive in this paper, compliments that for the probability distribution of avalanche temporal durations [8]. From the exact result $\gamma = 2$ it follows that the values of exponents τ and μ in the one-dimensional anisotropic BS model are related via $\tau = 2\mu$. Using the approximate form of the function $\tau(\mu)$, derived in [9], we can give an analytical estimate of the exponents τ and μ in a 1D anisotropic BS model. Indeed, they must lie at the intersection point of the $\tau = 2\mu$ line and the $\tau(\mu)$ curve, valid for an arbitrary BS model [9]. These analytical estimates are in excellent agreement with the direct numerical integration of the exact master equation for the probability distribution function of the avalanche spatial sizes, giving $\mu = 0.588(1)$ and $\tau = 1.176(2)$, as well as Monte Carlo simulations performed by us and by the authors of Refs. [12,13].

The scale-free stationary state of an arbitrary extremal model can be characterized by a number of critical exponents. Several scaling relations, reviewed and discussed in Ref. [3], reduce this variety to only two independent exponents, such as τ for the power law in the distribution $P(s) \sim s^{-\tau}$ of avalanche temporal durations s and the dynamic exponent μ. The latter exponent relates the avalanche temporal duration s to its spatial volume $V(s)$ as $V(s) \sim s^{\mu}$. Here the spatial volume is defined as the number of distinct sites updated at least once during this avalanche. In the notation of Ref. [3], $\mu = d/D$ (or $\mu = d_{\mathrm{cov}}/D$ if the set of updated (covered) sites is not compact, but instead forms a fractal of dimension d_{cov}).

The Bak-Sneppen model in an arbitrary dimension and with an arbitrary anisotropy has an additional simplification [3]: since variables are simply replaced with new random numbers and have no memory about their previous values, the dynamics within a single avalanche is totally independent from what happened before it started. This observation [3] enables one to simulate f avalanches (for definitions see [3]) for an arbitrary value of f, which can be above as well as below the critical point, without specifying variables at passive sites (those with $f_i > f$) prior to avalanche. Another im-

1063-651X/98/58(6)/7141(5)/$15.00 PRE 58

portant consequence of the absence of interavalanche memory is that in any variant of the BS model there are no correlations between sizes of subsequent avalanches. This statement is *not* a result of any kind of mean-field approximation. Rather it is a clear logical consequence of the dynamical rules of the Bak-Sneppen model.

In [8] one of us derived a master equation for the distribution $P(s,f)$ of avalanche durations s, valid for a general BS model. Let us recall briefly the sequence of logical steps leading to this equation. The starting point is the analysis of the signal (minimal number as a function of time) $f_{\min}(t)$ of the model using an auxiliary parameter f. The intersection of this signal with the horizontal line drawn at f identifies the sequence of f avalanches, following one another. In other words, if $f_{\min}(t)>f$, $f_{\min}(t+k)<f$ for $1\leqslant k<s$, and $f_{\min}(t+s)>f$, we say that an f avalanche of size (temporal duration) s has occurred [3]. The sequence of f avalanches is characterized by the distribution $P(s,f)$ of their sizes s. One can investigate how this distribution changes under an infinitesimal increase of f from f to $f+df$. The change occurs simply because when the horizontal line at f is lifted, some intersections with the signal $f_{\min}(t)$ disappear. This means that two consecutive f avalanches of (temporal) sizes s_1 and s_2 merge into a single $f+df$ avalanche of size s_1+s_2. The occurrence of this event implies that at least one of the $V_1 \sim s_1^{\mu}$ sites, updated during the course of the first avalanche, has $f<f_i<f+df$. Taking into account that, as we argued above, subsequent avalanches in the Bak-Sneppen model are uncorrelated, one can write the balance of loss and gain of f avalanches of size s as f is increased. The resulting master equation for the distribution $P(s,f)$ is given by

$$\partial_f P(s,f) = -s^{\mu} P(s,f) + \sum_{s_1=1}^{s-1} s_1^{\mu} P(s_1,f) P(s-s_1,f). \tag{1}$$

Strictly speaking, in order for the above equation to be *exact* one needs to replace s^{μ} with the average number of updated sites $V(s,f)$, where the average is taken over all f avalanches of size s. As was observed numerically, this quantity has an insignificant f dependence and its large s asymptotics is well described by the power law $V(s,f)\simeq As^{\mu}$. It was suggested in [8] and later convincingly confirmed numerically in [9] that the critical exponent μ uniquely determines the scaling properties of $P(s,f)$. This justifies our substitution of $V(s,f)$ by its asymptotical form s^{μ} in Eq. (1) (the constant A in front of s^{μ} was absorbed by redefinition of f).

The numerical integration of Eq. (1) with initial conditions $P(s,f=0)=\delta_{s,1}$ shows that as f approaches some critical value $f_c(\mu)$, $P(s,f)$ develops a power-law form with a diverging cutoff: $P(s,f)=s^{-\tau}F(s^{\sigma}(f_c-f))$. Above f_c there is a finite probability $p_{\infty}(f)$ to start an avalanche that never ends. The possibility of such an event shows up in Eq. (1) through the "normalization catastrophe," when $\Sigma_{s=1}^{\infty}P(s,f)$ for $f>f_c$ starts to fall below unity. This deviation is attributed to the appearance of the "infinite avalanche" with probability $p_{\infty}(f)\sim(f-f_c)^{\beta}$. This way the overall normalization $\Sigma_{s=1}^{\infty}P(s,f)+p_{\infty}(f)=1$ is satisfied at all f. The properties of Eq. (1) depend on the critical exponent μ. Given the value of this dynamic critical exponent, the remaining exponents τ, $\sigma=\mu+1-\tau$, $\gamma=(2-\tau)/\sigma$, and $\beta=(\tau-1)/\sigma$, as well as the scaling function $F(x)$ of the avalanche distribution near the critical point, follow from Eq. (1). In [9] the function $\tau(\mu)$ and the scaling form $F(x)$ were determined by numerical integration of Eq. (1) and the expansion around the mean-field point $\mu=1$, $\tau=3/2$. It was shown [9] that to second order in $1-\mu$ the function $\tau(\mu)$ is given by

$$\tau(\mu)=1.5-(1-\mu)+c(1-\mu)^2+O((1-\mu)^3),$$
$$c=\tfrac{4}{3}(\gamma_e+\ln 2-1)\simeq 0.3605. \tag{2}$$

Here $\gamma_e\simeq 0.5772$ is Euler's constant.

Our results for the one-dimensional anisotropic BS model are based on the exact equation for the probability distribution function $Q(r,f)$ of *spatial* sizes r of f avalanches. The derivation of this equation is very similar to the derivation of Eq. (1), briefly outlined above. The main difference lies in the fact that when an avalanche of spatial size r_1 merges with that of size r_2, they can overlap to form any spatial size between $\max(r_1,r_2)$ and r_1+r_2-1. Contrary to this, in the temporal domain the merging of avalanches of temporal durations s_1 and s_2 always produces an avalanche of temporal duration s_1+s_2. Let us analyze how $Q(r,f)$ changes when f is increased by an infinitesimal amount df. Some avalanches of size r will merge with the next avalanche. This event can only occur if one of the r sites, updated in the first avalanche, happens to host the smallest number right after the avalanche is finished. Each of these r sites has a variable f_i, which was randomly drawn from $\mathcal{P}(f)=e^{-f}$ during the course of the avalanche. At the end of this avalanche, by the very definition of an f avalanche, all these r sites have $f_i>f$. We can therefore regard the f_i's on these sites as randomly drawn from an exponential distribution normalized between f and ∞. The probability that a particular f_i is in the interval $[f,f+df]$, to linear order in df, is just df. The probability that at least one of the r sites has an f_i in the interval $[f,f+df]$ is $rdf+O(df^2)$. This implies that the number of f avalanches which will merge with the next one when f is raised to $f+df$ is $dQ(r)|_{\rm loss}=-r\,df\,Q(r,f)+O(df^2)$. Let us now consider a merging event between two f avalanches of size r_1 and r_2 resulting in an $f+df$ avalanche of size r. There are two scenarios of how this can happen. (i) The rightmost point of the second avalanche is at distance r from the leftmost (starting) point of the first avalanche; the constraint on possible values of r_1 and r_2 imposed by this scenario is $\max(r_1,r_2)\leqslant r\leqslant r_1+r_2-1$. (ii) $r_1=r$, and the second avalanche is fully contained within the first one. In the former case, the values of r, r_1, and r_2 uniquely specify the initial site of the second avalanche. Therefore, the probability of this event occurring is just the probability $df+O(df^2)$ that this site has $f_i\in[f,f+df]$. However, in the latter case the starting point of the second avalanche can be any of the first $r-r_2$ sites of the first avalanche. This event occurs with a probability $(r-r_2)df+O(df^2)$. Putting all these terms together, we find

$$\partial_f Q(r,f) = -rQ(r,f) + \sum_{r_1=1}^{r} Q(r_1,f) \sum_{r_2=r-r_1+1}^{r} Q(r_2,f) + Q(r,f)\sum_{r_2=1}^{r}(r-r_2)Q(r_2,f). \tag{3}$$

This is an *exact* equation for the distribution of spatial sizes of *f* avalanches. Unlike our previous results, its validity does not require any scaling assumptions, such as $V(s,f) \sim s^{\mu}$ used in the derivation of Eq. (1). Equation (3) has to be solved with the initial condition $Q(r,f=0)=\delta_{r,2}$. Indeed, in the one-dimensional anisotropic BS model at any time step (or $f=0$ avalanche for that matter), $r=2$ sites are updated.

A similar but more complicated equation can be written for the *isotropic* one-dimensional Bak-Sneppen model. The basic object of this equation is the probability distribution $Q(r_1,r_2,f)$ of *f* avalanches replacing precisely r_1 sites to the left of the starting point and r_2 sites to the right of it. The initial condition is given by $Q(r_1,r_2,0)=\delta_{r_1,1}\delta_{r_2,1}$. Although we were unable to derive any analytical exponent relations from this equation, it should be stressed that it contains all properties of the one-dimensional isotropic BS model, and in principle its numerical integration constitutes a viable alternative to Monte Carlo simulations of the model.

Yet another variant of Eq. (3) can be written for the anisotropic Bak-Sneppen model in dimensions higher than 1. Let r denote the spatial extent of the avalanche, measured along the diagonal $(1,1,\dots,1)$ of the d-dimensional space. In projection to this axis the starting point of an avalanche is always the leftmost point in the avalanche. In order to describe the shape of the region covered by the avalanche, one needs to introduce the new exponent ζ, defined by the average number of updated sites $n_{\mathrm{proj}}(r) \sim r^{\zeta}$ projected onto the same point at the distance r along the diagonal from the starting point. Then the total number of points covered by an avalanche of size r, $n_{\mathrm{cov}}(r) \sim n_{\mathrm{proj}}(r) r \sim r^{1+\zeta}$. In the 1D anisotropic BS model, $n_{\mathrm{proj}}(r)=1$, and, therefore, $\zeta=0$. By retracing arguments that led to the derivation of Eq. (3), it is easy to see that its higher-dimensional version can be written as

$$\partial_f Q(r,f) = -r^{1+\zeta}Q(r,f) + \sum_{r_1=1}^{r} r_1^{\zeta} Q(r_1,f) \sum_{r_2=r-r_1+1}^{r} Q(r_2,f) + Q(r,f)\sum_{r_s=1}^{r} r_s^{\zeta} \sum_{r_2=1}^{r-r_s-1} Q(r_2,f). \tag{4}$$

The drawback of this equation is that, similar to Eq. (1), it requires the input of an additional parameter (critical exponent) ζ.

Note that Eq. (3) for $Q(r,f)$ involves only $Q(r',f)$ for $r' \leqslant r$. Therefore, in principle this distribution can be computed numerically for $r \leqslant R$ to the desired accuracy. Forward numerical integration of Eq. (3) shows that as $f \to f_c \approx 1.2865$, $Q(r,f)$ develops a power-law behavior with an exponent $\tau_r = 1.299(3)$ (see Fig. 1). It is easy to see that in one dimension the exponent τ_r of the distribution of avalanche spatial sizes is related to the more familiar exponent τ of the distribution of their temporal durations through $\tau_r = (\tau-1)/\mu + 1$. Indeed, since asymptotically $r = As^{\mu}$, $s_0^{1-\tau} \sim P(s>s_0) = P(r>As_0^{\mu}) \sim (As_0^{\mu})^{1-\tau_r}$. Comparing powers of s_0 in this expression, one gets the above exponent relation. To find values of critical exponents hidden in Eq. (3), we study the behavior of moments $\langle r^n \rangle$ of the distribution $Q(r,f)$ as a function of *f*. These satisfy the equation

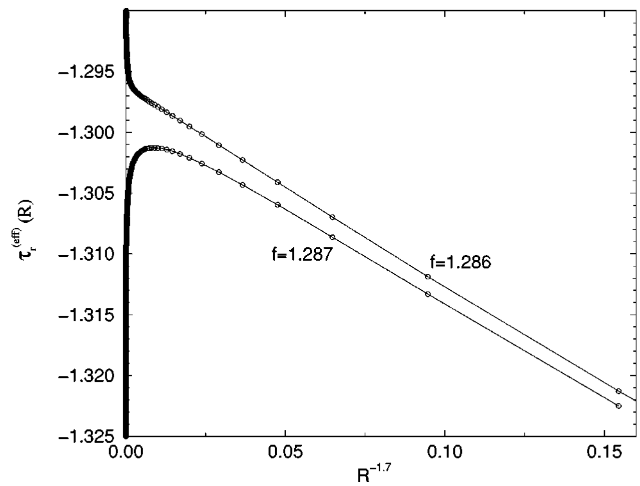

FIG. 1. The effective power-law exponent $\tau_r^{(\mathrm{eff})}(R)$, defined as $\tau_r^{(\mathrm{eff})}(R) = [\log_{10}Q(R,f) - \log_{10}Q(R-1,f)]/[\log_{10}R - \log_{10}(R-1)]$ for two different *f*. Since our method is free from finite-size effects, one can be sure that f_c is in (1.286,1.287) and $\tau_r = 1.299(3)$. $Q(R,f)$ was obtained by numerical integration of Eq. (3) with $R \leqslant R_{\max} = 2^{14} = 16\,384$. A second-order Runge-Kutta method with $\delta f = 10^{-3}$ was used.

$$\partial_f \langle r^n \rangle = -\langle r^{n+1} \rangle + \sum_{r_+=1}^{\infty} \sum_{r_-=1}^{r_+} \left[2 \sum_{\rho=r_+}^{r_+ + r_- - 1} \rho^n + r_+^n (r_+ - r_-) \right] Q(r_+,f) Q(r_-,f), \tag{5}$$

where the sum over r_1 and r_2 has been transformed into a sum over $r_+ = \max(r_1,r_2)$ and $r_- = \min(r_1,r_2)$. For $n=1$, Eq. (5) reads $\partial_f \langle r \rangle = -\langle r^2 \rangle + \Sigma_{r_+=1}^{\infty}\Sigma_{r_-=1}^{r_+}(r_+^2 + r_-^2 + r_- r_+ - r_-)Q(r_+,f)Q(r_-,f) = -\langle r^2 \rangle + \Sigma_{r_1=1}^{\infty}\Sigma_{r_2=1}^{\infty}[r_1^2 + r_2^2 + r_1 r_2 - \min(r_1,r_2)]/2 Q(r_1,f)Q(r_2,f)$. For $f<f_c$, when there are no infinite avalanches and the avalanche distribution $Q(r,f)$ is normalized to unity, one gets

$$\partial_f \langle r \rangle = \frac{\langle r \rangle^2}{2} - \frac{\langle \min(r_1,r_2) \rangle}{2}. \tag{6}$$

Close to the critical point we can neglect the term $\langle \min(r_1,r_2) \rangle$, since it diverges slower than $\langle r \rangle^2$ and we are left with $\partial_f \langle r \rangle \simeq \langle r \rangle^2/2$. This has an obvious solution

$$\langle r \rangle = \frac{2}{f_c - f} + O\left(\frac{1}{(f_c-f)^2} \right). \tag{7}$$

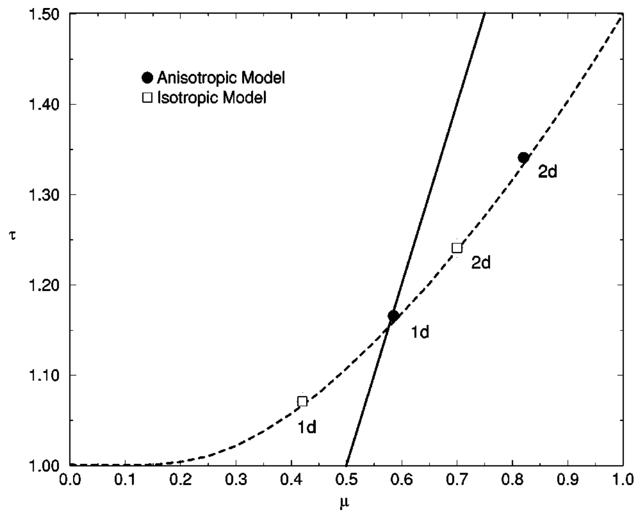

FIG. 2. The curve $\tau(\mu)$ from Ref. [9] is shown along with the results of Monte Carlo simulations of both isotropic (○) and anisotropic (●) BS models in one and two dimensions. The intersection point of $\tau(\mu)$ with the line $\tau=2\mu$ determines the exponents of the 1D anisotropic BS model in agreement with Monte Carlo results.

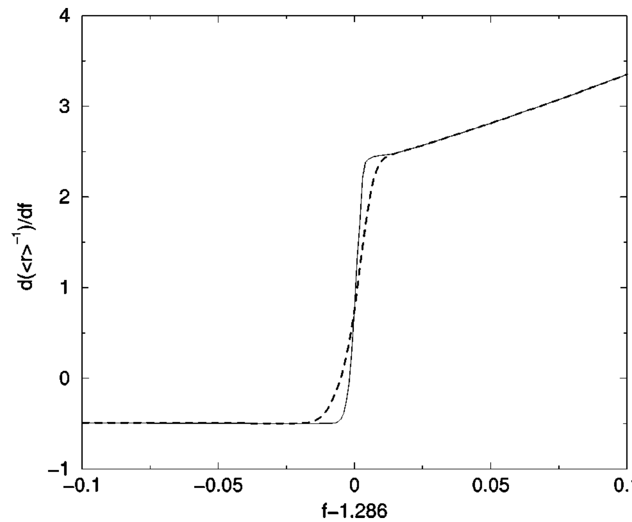

FIG. 3. The plot of $d(\langle r\rangle^{-1})/df$ vs f from the numerical integration of Eq. (3) for $r\leqslant R=2^{12}=1024$ (dashed line) and $r\leqslant R=2^{14}=16\,384$ (solid line). A second-order Runge-Kutta method with $\delta f=10^{-3}$ was used.

It has been known for some time that in the Bak-Sneppen model as well as in several other extremal models, the *amplitude* of the divergence of $\langle r\rangle$ as $f\to f_c^-$ is given by the critical exponent γ. This exponent describes the critical behavior of the average avalanche duration as $\langle s\rangle\sim|f_c-f|^{-\gamma}$. For the original derivation of this fact [14], see Eq. (17) in [3]. Actually, as was shown in [8], this fact can also be derived from Eq. (1). Indeed, from this equation it is easy to see that in the absence of an infinite avalanche the first moment of $P(s,f)$ obeys $\partial_f\langle s\rangle=\langle s^\mu\rangle\langle s\rangle$. Therefore, in the critical region one has $\langle r\rangle=\langle s^\mu\rangle=\partial_f(\ln\langle s\rangle)=\gamma/(f_c-f)+O((f_c-f)^{-2})$! From Eq. (7) we conclude that in the anisotropic one-dimensional Bak-Sneppen model

$$\gamma=2. \tag{8}$$

Using the scaling relation [3] $\gamma=(2-\tau)/\sigma=(2-\tau)/(1+\mu-\tau)$, we readily find that Eq. (8) implies $\tau=2\mu$. This, combined with the $\tau(\mu)$ relation found in [9], gives $\mu=0.58(1)$ and $\tau=1.16(2)$ as coordinates of the intersection point (see Fig. 2). The uncertainty in these numbers comes from our approximate knowledge of the systematic errors in the $\tau(\mu)$ curve, determined by numerical integration of Eq. (1). Presently, this integration was performed and an approximate power-law exponent τ was measured for $P(s,f)$ with $s\leqslant 2^{14}\simeq 1.6\times 10^4$. To improve the precision we can use a very accurate value for $\tau_r=1+(\tau-1)/\mu=1.299(3)$ measured by direct numerical integration of Eq. (3). Our resources allowed us to integrate this equation forward for $r\leqslant 2^{14}$, which is equivalent to measuring $P(s,f)$ up to $s=r^{1/\mu}\simeq 1.5\times 10^7$, i.e., over a much wider range than from numerical integration of Eq. (1). Using exponent relations $\tau=2\mu$ and $\tau=1+\mu(\tau_r-1)$ from $\tau_r=1.299(3)$, one gets

$$\tau=1.176(2), \tag{9}$$

$$\mu=0.588(1). \tag{10}$$

These are our best estimates of two basic exponents of the one-dimensional anisotropic BS model. The Monte Carlo simulations of the anisotropic BS model in $d=1$ are in perfect agreement with these values for τ and μ. Indeed, Head and Rodgers [13] found $\mu=0.59(3)$, while in [12] it was measured to be $\mu=0.60(1)$. In [13] they also measured the exponent $\pi=2.42(5)$ of the distribution of spatial jumps of the minimal site. This should be compared to our prediction for this exponent based on the scaling relation $\pi=1+(2-\tau)/\mu=2.401(6)$. We have also performed Monte Carlo simulations of the anisotropic BS model in one and two dimensions. In 1D we found $\mu_{1D}=0.58(1)$ and $\tau_{1D}=1.17(1)$ in agreement with [12,13], our analytical results, and the direct simulation of Eq. (3). In two dimensions our Monte Carlo simulations give $\mu_{2D}=0.83(1)$ and $\tau_{2D}=1.35(1)$. As shown in Fig. 2, these exponents lie on the $\tau(\mu)$ curve, valid for a general BS model. However, in the 2D anisotropic BS model we do not know the exact value of γ to fix the position of the exponents on this curve.

We also tested our theoretical prediction $\langle r\rangle\simeq 2/(f_c-f)$ (which led us to $\gamma=2$) against the direct numerical integration of Eq. (3). We measured the first moment $\langle r\rangle$ of the distribution $Q(r,f)$, obtained as described above by direct numerical integration of Eq. (3). As expected, the power-law divergence of $\langle r\rangle$ both above and below f_c has the exponent -1. It can be clearly seen when the numerical derivative $d(\langle r\rangle^{-1})/df$ is plotted as a function of f (see Fig. 3). This derivative approaches different *finite* limits as $f\to f_c\pm 0$. The divergence of $\langle r\rangle$ at f_c is clearly cut off by the finite size R of avalanches considered. This is illustrated in Fig. 3 by showing two curves for two different cutoff sizes $R=2^{12}$ and 2^{14}. From the asymptotical value of $d(\langle r\rangle^{-1})/df$ as $f\to f_c-0$, we measure $1/\gamma=0.498(3)$, which yields $\gamma=2.01(2)$, in complete agreement with Eq. (8).

We were able to estimate yet another critical exponent from this plot. As was shown in [8], the divergence of $\langle r\rangle$ in the overcritical regime *above* f_c is given by $\langle r\rangle=\beta/(f-f_c)$, where $\beta=(\tau-1)/(1+\mu-\tau)$ is the ''order parameter'' exponent describing the scaling of the probability to start an infinite avalanche $p_\infty(f)\sim(f-f_c)^\beta$. For $p_\infty=1-\Sigma_{s=1}^\infty P(s,f)$, Eq. (1) readily gives $\partial_f p_\infty=\langle s^\mu\rangle p_\infty$. Therefore, above the critical point one has $\langle r\rangle=\langle s^\mu\rangle=\partial_f(\ln p_\infty)=\beta/(f-f_c)+O((f-f_c)^{-2})$! The quadratic fit to $d(\langle r\rangle^{-1})/df$ above f_c gives $1/\beta=2.34(2)$ or $\beta=0.427(4)$. This is in excellent agreement with our theoretical prediction $\beta=(\tau-1)/(1+\mu-\tau)=0.427(6)$ based on the best estimate of $\tau_r=1.299(3)$ and the exponent relation $\beta=(\tau_r-1)/(2-\tau_r)$.

In summary, we have analyzed the behavior of the anisotropic Bak-Sneppen model. We demonstrated that a nontrivial relation between critical exponents τ and μ, recently derived for the isotropic Bak-Sneppen model [8,9], holds for its anisotropic version as well. The exponents measured by Monte Carlo simulations of the anisotropic Bak-Sneppen model in one and two dimensions are in agreement with this relation. For the one-dimensional anisotropic Bak-Sneppen model we derive a novel exact equation (3) for the distribution $Q(r,f)$ of avalanche spatial sizes. We also propose analogous equations for the one-dimensional isotropic BS model and the anisotropic BS model in $d>1$. By studying the behavior of the first moment of the distribution $Q(r,f)$, we managed to extract the exact value $\gamma=2$ for one of the critical exponents of the one-dimensional anisotropic BS model. The values of the critical exponents τ and $\mu=d/D$ were found as coordinates of the intersection point between $\tau(\mu)$ and $\gamma(\tau,\mu)=2$ curves. They are in excellent agreement with both Monte Carlo simulations of the model as well as results of numerical integration of the master equation for $Q(r,f)$. We summarize our best estimates for the exponents in one and two dimensions in Table I.

TABLE I. The results from this table are obtained by numerical integration of Eq. (3) for $d=1$ and Monte Carlo simulations for $d=2$. The existing exact exponent relations were then applied.

Exponent	1D anisotropic BS	2D anisotropic BS
τ	1.176(2)	1.35(1)
μ	0.588(1)	0.83(1)
σ	0.412(1)	0.48(2)
γ	2	1.35(4)
β	0.427(6)	0.73(4)
D	1.701(3)	2.41(3)
π	2.401(6)	2.57(3)

One of us (S.M.) would like to thank N. D. Mermin for asking the question that triggered this study. The work at Brookhaven National Laboratory was supported by the U.S. Department of Energy, Division of Material Science under Contract No. DE-AC02-98CH10886.

[1] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).
[2] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[3] M. Paczuski, S. Maslov, and P. Bak, Phys. Rev. E **53**, 414 (1996).
[4] H. Flyvbjerg, K. Sneppen, and P. Bak, Phys. Rev. Lett. **71**, 4087 (1993); J. de Boer, B. Derrida, H. Flyvbjerg, A. D. Jackson, and T. Wettig, *ibid.* **73**, 906 (1994).
[5] M. Marsili, Europhys. Lett. **28**, 385 (1994); B. Mikeska, Phys. Rev. E **55**, 3708 (1997).
[6] S. Maslov, Phys. Rev. Lett. **74**, 562 (1995).
[7] S. Boettcher and M. Paczuski, Phys. Rev. Lett. **76**, 348 (1996).
[8] S. Maslov, Phys. Rev. Lett. **77**, 1182 (1996).
[9] M. Marsili, P. De Los Rios, and S. Maslov, Phys. Rev. Lett. **80**, 1457 (1998).
[10] P. De Los Rios, M. Marsili, and M. Vendruscolo, Phys. Rev. Lett. **80**, 5746 (1998).
[11] The BS model is ''insensitive'' to the shape of random number distribution because only the *order* among variables f_i matters in the dynamics. With the rescaling map $f(f')=-\ln\int_{f'}^\infty P(f')df'$, the BS dynamics for a general $P(f')$ is turned into that with the exponential distribution.
[12] M. Vendruscolo, P. De Los Rios, and L. Bonesi, Phys. Rev. E **54**, 6053 (1996).
[13] D. A. Head and G. J. Rodgers, J. Phys. A **31**, 3977 (1998) (LANL e-print cond-mat/9802028).
[14] Traditionally the Bak-Sneppen model was studied with $\mathcal{P}(f)$ being a flat distribution on [0,1]. In this case one has $\langle r\rangle=\gamma(1-f_c)/(f_c-f)+O((f_c-f)^{-2})$. After rescaling of f_i as described in [11] one recovers our result for $\mathcal{P}(f)=e^{-f}$.